

THE PICARD GROUP OF MOTIVIC $\mathcal{A}(1)$

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ABSTRACT. We show that the Picard group $\mathrm{Pic}(\mathcal{A}(1))$ of the stable category of modules over \mathbb{C} -motivic $\mathcal{A}(1)$ is isomorphic to \mathbb{Z}^4 . By comparison, the Picard group of classical $\mathcal{A}(1)$ is $\mathbb{Z}^2 \oplus \mathbb{Z}/2$. One extra copy of \mathbb{Z} arises from the motivic bigrading. The joker is a well-known exotic element of order 2 in the Picard group of classical $\mathcal{A}(1)$. The \mathbb{C} -motivic joker has infinite order.

1. INTRODUCTION

1.1. The Picard group of $\mathcal{A}(1)$. Let $\mathcal{A}(1)^{\mathrm{cl}}$ be the subalgebra of the classical mod 2 Steenrod algebra generated by Sq^1 and Sq^2 . The stable module category $\mathrm{Stab}(\mathcal{A}(1)^{\mathrm{cl}})$ is the category whose objects are the finitely generated graded left $\mathcal{A}(1)^{\mathrm{cl}}$ -modules, and whose morphisms are the usual $\mathcal{A}(1)^{\mathrm{cl}}$ -module maps, modulo maps that factor through projective $\mathcal{A}(1)^{\mathrm{cl}}$ -modules.

The stable module category $\mathrm{Stab}(\mathcal{A}(1)^{\mathrm{cl}})$ is equipped with a tensor product over \mathbb{F}_2 . The unit of this pairing is \mathbb{F}_2 , and an object M of $\mathrm{Stab}(\mathcal{A}(1)^{\mathrm{cl}})$ is invertible if there exists another $\mathcal{A}(1)^{\mathrm{cl}}$ -module N such that $M \otimes_{\mathbb{F}_2} N$ is stably isomorphic to \mathbb{F}_2 . The Picard group $\mathrm{Pic}(\mathcal{A}(1)^{\mathrm{cl}})$ is the set of invertible stable isomorphism classes, with group operation given by tensor product over \mathbb{F}_2 .

Ext groups over $\mathcal{A}(1)^{\mathrm{cl}}$ are invariants of stable isomorphism classes of $\mathcal{A}(1)^{\mathrm{cl}}$ -modules. Thus, $\mathrm{Stab}(\mathcal{A}(1)^{\mathrm{cl}})$ is the natural category on which Ext groups over $\mathcal{A}(1)^{\mathrm{cl}}$ are defined. These Ext groups are of topological interest because of the Adams spectral sequence

$$E_2 = \mathrm{Ext}_{\mathcal{A}(1)^{\mathrm{cl}}}(H\mathbb{F}_2^*(X), \mathbb{F}_2) \Rightarrow ko_*(X)_2^\wedge$$

converging to 2-completed ko -homology.

Adams and Priddy computed $\mathrm{Pic}(\mathcal{A}(1)^{\mathrm{cl}})$ while studying infinite loop space structures on the classifying space BSO [AP76, Section 3]. They found that the Picard group is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2$. One copy of \mathbb{Z} comes from the grading; one can shift the grading on $\mathcal{A}(1)^{\mathrm{cl}}$ -modules to obtain “new” $\mathcal{A}(1)^{\mathrm{cl}}$ -modules. The other copy of \mathbb{Z} comes from the algebraic loop functor that is a formal part of the stable module category; see Definition 2.13 below for more details.

The copy of $\mathbb{Z}/2$ in $\mathrm{Pic}(\mathcal{A}(1)^{\mathrm{cl}})$ is the most interesting part of the calculation. It is exotic in the sense that it doesn’t follow from the formal theory of stable module categories and Picard groups. The copy of $\mathbb{Z}/2$ is generated by the joker J shown in Figure 3. It turns out that $J \otimes_{\mathbb{F}_2} J$ is stably isomorphic to \mathbb{F}_2 , so J has order 2 in $\mathrm{Pic}(\mathcal{A}(1)^{\mathrm{cl}})$.

1.2. The motivic setting. There has been much recent work on the computational side of motivic homotopy theory. In particular, the algebraic properties of the motivic Steenrod algebra have come under close scrutiny. As part of this program, it is natural to ask about the Picard group of the motivic version of $\mathcal{A}(1)$.

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The goal of this article is to carry out this computation for \mathbb{C} -motivic $\mathcal{A}(1)$, which is the simplest motivic case.

The fundamental difficulty in the motivic situation is that the ground ring \mathbb{M}_2 is not a field. Rather, it is a graded polynomial ring $\mathbb{F}_2[\tau]$. Therefore, we must be careful to insert \mathbb{M}_2 -freeness hypotheses at the appropriate places.

We will show that $\text{Pic}(\mathcal{A}(1))$ is isomorphic to \mathbb{Z}^4 . Two copies of \mathbb{Z} arise from the motivic bigrading, and one copy of \mathbb{Z} comes from the algebraic loop functor. This leaves one copy of \mathbb{Z} , which is generated by the motivic joker J (see Figure 3). It turns out that the motivic joker has infinite order. The order of the motivic joker is the essential new aspect of the motivic calculation.

There are two main ideas in the proof. First, the Hopf algebra $\mathcal{A}(1)/\tau$ is isomorphic to the group algebra of the dihedral group D_8 of order 8, so $\mathcal{A}(1)/\tau$ is well-understood. In particular, the Picard group of $\mathcal{A}(1)/\tau$ is known.

Second, consider the functor that takes an $\mathcal{A}(1)$ -module M to its quotient M/τ . In general, quotienting is not an exact functor. However, it turns out to be well-behaved for $\mathcal{A}(1)$ -modules that are \mathbb{M}_2 -free. Using this well-behaved functor, we can pull back information about the Picard group of $\mathcal{A}(1)/\tau$ to information about the Picard group of $\mathcal{A}(1)$.

The difference between the \mathbb{C} -motivic and classical Picard groups is a familiar one. Frequently, motivic computations are larger than classical ones. However, they are also often more regular. This situation is clearly displayed in our work, where the motivic Picard group is free, while the classical Picard group has torsion.

We do not consider the Picard group of motivic $\mathcal{A}(1)$ over other base fields. The \mathbb{C} -motivic phenomena described in this paper will occur over other base fields, but it is possible that additional complications arise.

Our computation of the Picard group of motivic $\mathcal{A}(1)$ is potentially useful for the following problem. From our perspective, the most essential property of the \mathbb{C} -motivic spectrum ko is that its cohomology is isomorphic to $\mathcal{A}/\mathcal{A}(1)$ [IS11]. One might ask whether such a \mathbb{C} -motivic spectrum is unique. Suppose that X and Y are \mathbb{C} -motivic spectra whose cohomology modules are both isomorphic to $\mathcal{A}/\mathcal{A}(1)$. In order to construct an equivalence between X and Y , one could compute the maps between X and Y via the motivic Adams spectral sequence, whose E_2 -page takes the form $\text{Ext}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}(1), \mathcal{A}/\mathcal{A}(1))$. By a standard change of rings theorem, this E_2 -page is equal to $\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathcal{A}/\mathcal{A}(1))$. It is possible that this Adams spectral sequence is analyzable, because $\mathcal{A}/\mathcal{A}(1)$ probably splits as an $\mathcal{A}(1)$ -module into summands that belong to the Picard group. We leave the details for future work.

2. STABLE MODULE THEORY OF FINITE MOTIVIC HOPF ALGEBRAS

2.1. Finite motivic Hopf algebras. We use the same notation and framework as in [Isa14]. We work in the \mathbb{C} -motivic setting at the prime 2. The base ring is the motivic cohomology $H^{*,*}(S^{0,0}; \mathbb{F}_2)$ of the sphere spectrum. We write \mathbb{M}_2 for this ring; it is isomorphic to $\mathbb{F}_2[\tau]$ with τ in bidegree $(0, 1)$. Objects are bigraded in the form (s, w) , where s corresponds to the classical internal degree and w is the motivic weight.

Let \mathcal{A} be the \mathbb{C} -motivic Steenrod algebra at the prime 2. This Hopf algebra over \mathbb{M}_2 was first computed in [Voe03], and its structure is thoroughly understood.

A fundamental difference between the \mathbb{C} -motivic and the classical situations is that the base ring \mathbb{M}_2 is not a field. Therefore, we must add freeness over \mathbb{M}_2 as a hypothesis in Definition 2.1 below.

Definition 2.1. A *finite motivic Hopf algebra* is a cocommutative bigraded Hopf algebra over \mathbb{M}_2 that is finitely generated and free as an \mathbb{M}_2 -module.

Example 2.2. Recall the subalgebras $\mathcal{A}(n)^{\text{cl}}$ and $\mathcal{E}(n)^{\text{cl}}$ of the classical Steenrod algebra [AM74]. These subalgebras have \mathbb{C} -motivic analogues, and they are finite motivic Hopf algebras.

Throughout the article, A will represent an arbitrary finite motivic Hopf algebra, while \mathcal{A} represents the \mathbb{C} -motivic Steenrod algebra. Note that \mathcal{A} is not finitely generated as an \mathbb{M}_2 -module. However, we are primarily interested in the subalgebra $\mathcal{A}(1)$ of \mathcal{A} generated by Sq^1 and Sq^2 , and $\mathcal{A}(1)$ is a finitely generated \mathbb{M}_2 -module.

Lemma 2.3. *Suppose that A is a finite motivic Hopf algebra. If M is a finitely generated projective A -module, then it is a finitely generated free \mathbb{M}_2 -module.*

Proof. Suppose that M is a finitely generated projective A -module. Then M is a summand of free A -module F . The module F is free and finitely generated as an \mathbb{M}_2 -module, since A is free and finitely generated as an \mathbb{M}_2 -module. Therefore, as an \mathbb{M}_2 -module, M is a summand of a free \mathbb{M}_2 -module. This shows that M is a finitely generated projective \mathbb{M}_2 -module.

It remains to show that finitely generated projective \mathbb{M}_2 -modules are free. The ring \mathbb{M}_2 is a graded principal ideal domain whose graded ideals are of the form (τ^k) . Therefore, a finitely generated \mathbb{M}_2 -module is a direct sum of a free module and cyclic modules of the form \mathbb{M}_2/τ^k . It follows that finitely generated projective \mathbb{M}_2 -modules are the same as finitely generated free \mathbb{M}_2 -modules. \square

2.2. The stable category. We now recall the basic framework of stable module categories, as applied to a finite motivic Hopf algebra A . The stable category of modules over a group algebra is a classical construction in group representation theory [CTVEZ03, Section 2.6]. In the case of a finite motivic Hopf algebra A , the theory is similar to the case when A is a finite dimensional graded connected Hopf algebra over a field, for which a good reference is [Mar83, Section 14.1]. However, since the base ring \mathbb{M}_2 of a finite motivic Hopf algebra is not a field, one has to pay attention to the underlying theory of \mathbb{M}_2 -modules, and add \mathbb{M}_2 -freeness hypotheses when appropriate.

Definition 2.4. Let ${}_A\mathbf{Mod}$ be the category of bigraded finitely generated left A -modules, and let ${}_A\mathbf{Mod}^f$ be the full subcategory of ${}_A\mathbf{Mod}$ consisting of left A -modules that are free over \mathbb{M}_2 .

Definition 2.5. Let $\text{Stab}(A)$ be the category whose objects are the same as in ${}_A\mathbf{Mod}^f$, and whose morphisms are given by

$$\text{Hom}_{\text{Stab}(A)}(M, N) = \text{Hom}_A(M, N) / \sim,$$

where two morphisms f and g are equivalent if their difference factors through a projective A -module.

If M and N are objects of ${}_A\mathbf{Mod}^f$, then we write $M \cong N$ if M and N are stably equivalent, i.e., if they are isomorphic in the stable category $\text{Stab}(A)$.

Our main interest is the Picard group $\text{Pic}(A)$ of the stable category of some finite motivic Hopf algebra A . We will see below in Remark 2.9 that all representatives of every element in $\text{Pic}(A)$ are actually free over \mathbb{M}_2 and thus captured by $\text{Stab}(A)$. In other words, the assumptions about \mathbb{M}_2 -freeness in Definitions 2.4 and 2.5 are no loss of generality.

In the same vein, it is essential that we use constructions that preserve \mathbb{M}_2 -free A -modules. For example for any finitely generated A -module M (not necessarily \mathbb{M}_2 -free), the algebraic loop ΩM (defined below in Definition 2.13) is free over \mathbb{M}_2 , as it is the kernel of a map from a finitely generated free \mathbb{M}_2 -module.

The stable category $\text{Stab}(A)$ is naturally enriched over A -modules, since the equivalence relation on morphisms is A -linear. The category $\text{Stab}(A)$ has additional structure that we describe next.

Proposition 2.6. *The category $\text{Stab}(A)$ is a closed symmetric monoidal category.*

Proof. This is a standard result from the theory of stable modules; see [Mar83, Proposition 15.2.19] for example. The only additional observation is that the tensor product of \mathbb{M}_2 -free modules is \mathbb{M}_2 -free. \square

2.3. Picard groups.

Definition 2.7. Let A be a finite motivic Hopf algebra. The Picard group $\text{Pic}(A)$ is the group (of isomorphism classes) of invertible objects of $\text{Stab}(A)$ under the monoidal structure, *i.e.*, the group of stably invertible modules with the tensor product as group law.

Note that $\text{Pic}(A)$ is an abelian group because $\text{Stab}(A)$ is symmetric monoidal.

Remark 2.8. In Definition 2.7, we are only considering finitely generated A -modules. This is no loss of generality because every invertible object must be finitely generated. This follows from [MS14, Proposition 2.1.3], for example.

Remark 2.9. In Definition 2.7, we have defined the Picard group using only A -modules that are \mathbb{M}_2 -free. In fact, if M and N are arbitrary finitely generated A -modules such that $M \otimes N$ is stably equivalent to \mathbb{M}_2 , then M and N must in fact be \mathbb{M}_2 -free. In other words, there is no harm in considering only \mathbb{M}_2 -free modules in the Picard group. For if $M \otimes N$ is isomorphic to $\mathbb{M}_2 \oplus P$ for some projective A -module P , then P is \mathbb{M}_2 -free by Lemma 2.3. Therefore, $M \otimes N$ is \mathbb{M}_2 -free, and M and N are \mathbb{M}_2 -free as well.

Definition 2.10. Denote the \mathbb{M}_2 -linear dual functor by

$$D : {}_A\mathbf{Mod}^{\text{op}} \longrightarrow {}_A\mathbf{Mod} : M \longmapsto DM = \text{Hom}_{\mathbb{M}_2}(M, \mathbb{M}_2).$$

Lemma 2.11. *The \mathbb{M}_2 -linear dual functor D induces a functor*

$$D : \text{Stab}(A)^{\text{op}} \longrightarrow \text{Stab}(A).$$

Proof. The dual functor D preserves \mathbb{M}_2 -freeness because D is defined as Hom over \mathbb{M}_2 .

It suffices to check that if P is A -projective, then DP is A -projective. Since the dual respects direct sums, it is enough to show that DA is projective. This follows as in [Mar83, Theorem 12.2.9] by considering a retraction

$$DA \longrightarrow DA \otimes A \longrightarrow DA$$

and observing that the “shearing map” [Mar83, Proposition 12.1.4] makes $DA \otimes A$ into a free A -module. \square

Lemma 2.12 shows that the dual functor D corresponds to inversion in the Picard group.

Lemma 2.12. *Let M be an A -module. The evaluation morphism $DM \otimes M \xrightarrow{ev} \mathbb{M}_2$ is a stable equivalence if and only if M is invertible. In particular, the inverse of any element $[M]$ in $\text{Pic}(A)$ is its dual $[DM]$.*

Proof. This fact is standard in stable module theory; see [HPS97, Proposition A.2.8]. \square

We next describe the “algebraic loop” functor that is part of the structure of a stable module category.

Definition 2.13. Let Ω be the endo-functor of $\text{Stab}(A)$ given by

$$\Omega M = \ker(P \longrightarrow M)$$

where $P \longrightarrow M$ is any projective cover of M .

For $k \geq 0$, define $\Omega^k M$ inductively to be $\Omega(\Omega^{k-1} M)$.

For $k < 0$, define $\Omega^k M$ to be $D(\Omega^{-k} DM)$.

An immediate application of Schanuel's lemma shows that ΩM is independent of the choice of P . Note that ΩM is \mathbb{M}_2 -free because it is a subobject of P , and P is \mathbb{M}_2 -free by Lemma 2.3.

Lemma 2.14. *If M is stably invertible, then so is ΩM .*

Proof. This is a standard part of the theory of stable modules. It follows from the fact that $\Omega M \cong \Omega \mathbb{M}_2 \otimes M$, and that $\Omega \mathbb{M}_2$ is stably invertible; see [Bru12, Proposition 2.10] for example. \square

Lemma 2.14 implies that there is a group homomorphism

$$\eta : \mathbb{Z}^3 \longrightarrow \text{Pic}(A)$$

sending (m, n, s) to the stable class of $\Sigma^{m,n} \Omega^s \mathbb{M}_2$. This homomorphism constructs many elements in the Picard group of A . Such elements exist for essentially formal reasons and do not really reflect on the structure of the underlying algebra A . In a sense, the image of η consists of "uninteresting" invertible elements.

3. τ QUOTIENTS

Suppose that A is a finite motivic Hopf algebra. Then $A/\tau = \mathbb{F}_2 \otimes_{\mathbb{M}_2} A$ is a Hopf algebra. Since A/τ is defined over a field \mathbb{F}_2 , it is generally easier to understand than A itself. We shall use a change of basis functor that relates our finite motivic Hopf algebra A to the Hopf algebra A/τ .

Proposition 3.1. *Tensoring with the \mathbb{M}_2 -module \mathbb{F}_2 induces a strongly monoidal functor*

$${}_A \mathbf{Mod}^f \xrightarrow{(-)/\tau} {}_{A/\tau} \mathbf{Mod}^f$$

that preserves exact sequences. This functor passes to the stable category and thus induces a strongly monoidal functor

$$\text{Stab}(A) \xrightarrow{(-)/\tau} \text{Stab}(A/\tau).$$

Proof. The unit \mathbb{M}_2 of the monoidal structure of ${}_A \mathbf{Mod}^f$ is sent to the unit \mathbb{F}_2 . The functor is strongly monoidal since

$$M/\tau \otimes N/\tau \cong M \otimes N/\tau;$$

this is just an application of commuting colimits.

Consider a short exact sequence in ${}_A \mathbf{Mod}^f$. The sequence is split exact on the underlying free \mathbb{M}_2 -modules. It is still split exact as a sequence of \mathbb{F}_2 -modules after tensoring with \mathbb{F}_2 . This shows that $(-)/\tau$ is exact.

The functor sends free A -modules to free A/τ -modules. By additivity, we conclude that it sends projective A -modules to projective A/τ -modules and thus descends to the stable categories. \square

Remark 3.2. Note that ${}_{A/\tau} \mathbf{Mod}^f = {}_{A/\tau} \mathbf{Mod}$, since the ground ring \mathbb{F}_2 is a field.

Remark 3.3. The functor ${}_A\mathbf{Mod}^f \xrightarrow{(-)/\tau} {}_{A/\tau}\mathbf{Mod}^f$ of Proposition 3.1 preserves exact sequences, but it is not an “exact functor” in the usual sense because ${}_A\mathbf{Mod}^f$ is not an abelian category. Namely, the cokernel of a map in ${}_A\mathbf{Mod}^f$ need not be \mathbb{M}_2 -free.

We now come to the first major result that will allow us to understand the stable module category of a finite motivic Hopf algebra A . Lemma 3.4 identifies projective A -modules in terms of their quotients by τ .

Lemma 3.4. *Let A be a finite motivic Hopf algebra, and let M be a finitely generated A -module that is \mathbb{M}_2 -free. The following conditions are equivalent:*

- (1) *M is projective as an A -module.*
- (2) *M/τ is projective as an A/τ -module.*
- (3) *M/τ is free as an A/τ -module.*

Proof. Note that A/τ is a Frobenius algebra since it is a finite dimensional Hopf algebra over the field \mathbb{F}_2 [Mar83, Theorem 12.2.9]. In particular, projective A/τ -modules and free A/τ -modules are the same. This shows that conditions (2) and (3) are equivalent.

Now suppose that M is a projective A -module. Then M/τ is a projective A/τ -module by Proposition 3.1. This shows that condition (1) implies condition (2).

To show that condition (3) implies condition (1), suppose that M/τ is a free A/τ -module. We will show that $\mathrm{Ext}_A^i(M, N)$ vanishes for all A -modules N . In fact, it suffices to assume that N is finitely generated, for $\mathrm{Hom}_A(M, -)$ commutes with filtered colimits since M is finitely generated, and filtered colimits are exact.

Since M/τ is free over A/τ , we have an A -free resolution of M/τ of the form

$$\cdots \longrightarrow 0 \longrightarrow \bigoplus A \xrightarrow{\tau} \bigoplus A \longrightarrow M/\tau.$$

Therefore, $\mathrm{Ext}_A^i(M/\tau, N)$ vanishes whenever $i \geq 2$ and N is any A -module.

Since M is \mathbb{M}_2 -free, we have a short exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M/\tau \longrightarrow 0.$$

This sequence induces a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_A^i(M, N) \xrightarrow{\tau} \mathrm{Ext}_A^i(M, N) \longrightarrow \mathrm{Ext}_A^{i+1}(M/\tau, N) \longrightarrow \cdots$$

for all A -modules N . Since $\mathrm{Ext}_A^{i+1}(M/\tau, N)$ is zero for $i \geq 1$ by the previous paragraph, we conclude that the map

$$\mathrm{Ext}_A^i(M, N) \xrightarrow{\tau} \mathrm{Ext}_A^i(M, N)$$

is surjective for $i \geq 1$.

Note that M and N are finitely generated as \mathbb{M}_2 -modules since they are finitely generated as A -modules, and A is a finitely generated \mathbb{M}_2 -module. This implies that $\mathrm{Ext}_A^i(M, N)$ vanishes in sufficiently low motivic weights. The surjectivity of multiplication by τ then implies that $\mathrm{Ext}_A^i(M, N)$ vanishes in all weights. This means that M is a projective A -module. \square

Lemma 3.5. *Let M and N be finitely generated A -modules that are also \mathbb{M}_2 -free, and let $f : M \longrightarrow N$ be a map such that $f/\tau : M/\tau \longrightarrow N/\tau$ is injective. Then f is also injective, and the cokernel of f is \mathbb{M}_2 -free.*

Proof. Suppose that x is an element of M such that $f(x) = 0$, and let \bar{x} be the corresponding element in M/τ . Then $(f/\tau)(\bar{x})$ is zero, so \bar{x} is also zero because f/τ is injective. Therefore, x equals τy for some y . Now $\tau f(y) = f(\tau y) = f(x) = 0$, so $f(y)$ is also zero since N is \mathbb{M}_2 -free.

This shows that the kernel of f consists of elements that are infinitely divisible by τ . Since M is finitely generated, the kernel must be zero.

Now consider the cokernel N/M of f . Since N/M is finitely generated, it suffices to consider the annihilator of τ in N/M . We will show that this annihilator is zero.

Let x be an element of N , and let \bar{x} be the element of N/M that it represents. Suppose that $\tau\bar{x}$ is zero. Then τx belongs to M . Since f/τ is injective and $(f/\tau)(\tau x)$ is zero, we conclude as in the first paragraph that τx equals τy for some y in M . Since N is \mathbb{M}_2 -free, it follows that x equals y . In particular, x belongs to M . In other words, \bar{x} is zero. \square

The strong monoidal exact functor

$$-/ \tau : \text{Stab}(A) \longrightarrow \text{Stab}(A/\tau)$$

of Proposition 3.1 induces a group homomorphism

$$V : \text{Pic}(A) \longrightarrow \text{Pic}(A/\tau).$$

Proposition 3.6. *The map $V : \text{Pic}(A) \longrightarrow \text{Pic}(A/\tau)$ is injective.*

Proof. Let M be a finitely generated A -module such that M is \mathbb{M}_2 -free, and suppose that $[M]$ in $\text{Pic}(A)$ belongs to the kernel of V . Equivalently, M/τ and \mathbb{F}_2 are stably equivalent A/τ -modules. Since A/τ is a finite dimensional Frobenius algebra over \mathbb{F}_2 , we can use [Mar83, Proposition 14.11] to see that M/τ is isomorphic to the direct sum of \mathbb{F}_2 and a free A/τ -module. In other words, M/τ is isomorphic to $\mathbb{F}_2 \oplus F/\tau$, where F is a free A -module. Let j be the injection $F/\tau \longrightarrow M/\tau$.

There is a commutative diagram

$$\begin{array}{ccc} M & \twoheadrightarrow & M/\tau \\ \uparrow i & & \uparrow j \\ F & \twoheadrightarrow & F/\tau, \end{array}$$

in which the dashed arrow exists because F is A -projective and $M \twoheadrightarrow M/\tau$ is a surjection. By Lemma 3.5, i is injective because j is injective.

We now compute the cokernel C of i . Lemma 3.5 implies that C is \mathbb{M}_2 -free. Then Proposition 3.1 says that C/τ is isomorphic to the cokernel of j , which is \mathbb{F}_2 by inspection. We conclude that C is isomorphic to \mathbb{M}_2 .

Thus, there is a short exact sequence

$$F \hookrightarrow M \twoheadrightarrow \mathbb{M}_2,$$

so $M \twoheadrightarrow \mathbb{M}_2$ is a stable equivalence and $[M]$ is trivial in $\text{Pic}(A)$. \square

4. THE FINITE MOTIVIC HOPF ALGEBRA $\mathcal{A}(1)$

In this section, we introduce the specific finite motivic Hopf algebra $\mathcal{A}(1)$ whose Picard group we will compute.

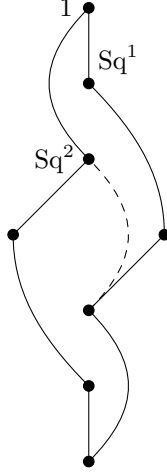
Definition 4.1. The finite motivic Hopf algebra $\mathcal{A}(1)$ is the \mathbb{M}_2 -subalgebra of the motivic Steenrod algebra generated by Sq^1 and Sq^2 .

Lemma 4.2. *The finite motivic Hopf algebra $\mathcal{A}(1)$ is isomorphic to*

$$\frac{\mathbb{M}_2[\text{Sq}^1, \text{Sq}^2]}{\text{Sq}^1 \text{Sq}^1, \text{Sq}^2 \text{Sq}^2 + \tau \text{Sq}^1 \text{Sq}^2 \text{Sq}^1, \text{Sq}^1 \text{Sq}^2 \text{Sq}^1 \text{Sq}^2 + \text{Sq}^2 \text{Sq}^1 \text{Sq}^2 \text{Sq}^1}.$$

The element Sq^1 is primitive, and $\Delta(\text{Sq}^2) = \text{Sq}^2 \otimes 1 + \tau \text{Sq}^1 \otimes \text{Sq}^1 + 1 \otimes \text{Sq}^2$.

Proof. This follows immediately from Voevodsky's description of the motivic Steenrod algebra [Voe03]. \square

FIGURE 1. The finite motivic Hopf algebra $\mathcal{A}(1)$

See Figure 1 for a picture of $\mathcal{A}(1)$. When writing $\mathcal{A}(1)$ -modules we use the following conventions. A straight line represents the action of Sq^1 , a curved line represents the action of Sq^2 , and a line is dotted if a squaring operation hits τ times a generator. For example, the dotted line in Figure 1 shows the relation $Sq^2 Sq^2 = \tau Sq^1 Sq^2 Sq^1$.

Lemma 4.3. *As ungraded Hopf algebras, $\mathcal{A}(1)/\tau$ is isomorphic to the group algebra $\mathbb{F}_2[D_8]$ of the dihedral group D_8 of order 8.*

Proof. Lemma 4.2 implies that $\mathcal{A}(1)/\tau$ is isomorphic to

$$\frac{\mathbb{F}_2[Sq^1, Sq^2]}{Sq^1 Sq^1, Sq^2 Sq^2, Sq^1 Sq^2 Sq^1 Sq^2 + Sq^2 Sq^1 Sq^2 Sq^1}.$$

For our purposes, a convenient presentation of D_8 consists of two generators x and y with the relations x^2 , y^2 , and $(xy)^4$. The isomorphism from $\mathcal{A}(1)/\tau$ to $\mathbb{F}_2[D_8]$ takes Sq^1 to $1 + x$ and Sq^2 to $1 + y$. \square

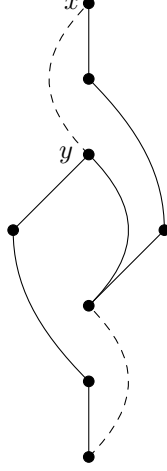
Recall that a sub-Hopf algebra B of a Hopf \mathbb{F}_2 -algebra A is elementary if it is isomorphic to an exterior algebra. Note that $Q_0 = Sq^1$ and $Q_1 = Sq^2 Sq^1 + Sq^1 Sq^2$ are elements of $\mathcal{A}(1)$ whose squares are zero.

Lemma 4.4. *The maximal elementary sub-Hopf algebras of $\mathcal{A}(1)/\tau$ are the exterior algebras $E(Q_0, Q_1)$ and $E(Sq^2, Q_1)$.*

Proof. Lemma 4.3 says that A/τ is isomorphic to the group algebra $\mathbb{F}_2[D_8]$ of the dihedral group of order 8. The elementary sub-Hopf algebras of $\mathbb{F}_2[D_8]$ correspond to the elementary abelian 2-subgroups of D_8 . The group D_8 has two maximal elementary abelian subgroups. Tracing back through the isomorphism of Lemma 4.3, one can identify the two maximal elementary sub-Hopf algebras of $\mathcal{A}(1)/\tau$. \square

4.1. Margolis homology. We now turn to an algebraic invariant detecting projectivity of $\mathcal{A}(1)$ -modules, analogous to Margolis's techniques using P_t^s -homology [Mar83].

Definition 4.5. Let x be an element of A such that x^2 is zero. For any A -module M , define the Margolis homology $H(M; x)$ to be the annihilator of x modulo the submodule xM .

FIGURE 2. The $\mathcal{A}(1)$ -module $\tilde{\mathcal{A}}(1)$

Recall that classically, an $\mathcal{A}(1)^{\text{cl}}$ -module M is projective if and only if $H(M; Q_0)$ and $H(M; Q_1)$ are both zero [AM71, Theorem 3.1], which is a direct consequence of a more general result [Pal97, Theorem 1.2-1.4]. Our goal is to generalize this result to the motivic situation.

Unfortunately, the motivic situation is more complicated. If M is an $\mathcal{A}(1)$ -module and $H(M; Q_0)$ and $H(M; Q_1)$ both vanish, then M is not necessarily projective.

Example 4.6. Let $\tilde{\mathcal{A}}(1)$ be the $\mathcal{A}(1)$ -module on two generators x and y of degrees $(0, 0)$ and $(2, 0)$ respectively, subject to the relations $\text{Sq}^2 x = \tau y$ and $\text{Sq}^1 \text{Sq}^2 \text{Sq}^1 x = \text{Sq}^2 y$. Figure 2 represents $\tilde{\mathcal{A}}(1)$ as an $\mathcal{A}(1)$ -module.

The Margolis homology groups $H(\tilde{\mathcal{A}}(1); Q_0)$ and $H(\tilde{\mathcal{A}}(1); Q_1)$ both vanish. However, $\tilde{\mathcal{A}}(1)$ is not a projective $\mathcal{A}(1)$ -module.

It turns out that we need two additional criteria for projectivity beyond Q_0 -homology and Margolis Q_1 -homology.

Proposition 4.7. *Let M be a finitely generated $\mathcal{A}(1)$ -module. Then M is projective if and only if:*

- (1) M is free over \mathbb{M}_2 ; and
- (2) $H(M/\tau; Q_0) = 0$; and
- (3) $H(M/\tau; Q_1) = 0$; and
- (4) $H(M/\tau; \text{Sq}^2) = 0$.

Proof. First suppose that M is projective. By inspection, conditions (2) through (4) are satisfied when M is $\mathcal{A}(1)$. Therefore, these conditions are satisfied when M is free. Using that a projective module is a summand of a free module, conditions (2) through (4) are also satisfied for any projective M . Finally, Lemma 2.3 shows that condition (1) is satisfied.

Now suppose that conditions (1) through (4) are satisfied. By Lemma 3.4, it suffices to show that M/τ is A/τ -projective. Note that A/τ -projectivity is detected by restriction to the quasi-elementary sub-Hopf algebras of A/τ [Pal97, Theorem 1.2-1.4]. See [Pal97, Definition 1.1] for the definition of quasi-elementary sub-Hopf algebras.

For group algebras, quasi-elementary sub-Hopf algebras coincide with elementary sub-Hopf algebras [Ser65] (as observed in [Pal97]). Since A/τ is isomorphic to the

group algebra $\mathbb{F}_2[D_8]$ by Lemma 4.3, Lemma 4.4 shows that the quasi-elementary sub-Hopf algebras of A/τ are the exterior algebra $E(Q_0, Q_1)$ and the exterior algebra $E(\text{Sq}^2, Q_1)$. Conditions (2) and (3) imply that M/τ is $E(Q_0, Q_1)$ -projective, and conditions (3) and (4) imply that M/τ is $E(\text{Sq}^2, Q_1)$ -projective. \square

Remark 4.8. The exterior algebra $E(Q_0, Q_1)$ is the unique maximal quasi-elementary sub-Hopf algebra of the classical Hopf algebra $\mathcal{A}(1)^{\text{cl}}$. This explains why condition (4) of Proposition 4.7 is absent from the classification of projective $\mathcal{A}(1)^{\text{cl}}$ -modules.

Corollary 4.9. *Let M and N be finitely generated $\mathcal{A}(1)$ -modules that are \mathbb{M}_2 -free, and let $f : M \rightarrow N$ be an $\mathcal{A}(1)$ -module map. Then f is a stable equivalence if and only if $f/\tau : M/\tau \rightarrow N/\tau$ induces an isomorphism in Margolis homologies with respect to Q_0 , Q_1 , and Sq^2 .*

Proof. We may choose a free $\mathcal{A}(1)$ -module F and a surjective map $g : M \oplus F \rightarrow N$ that restricts to f on M . Then f is a stable equivalence if and only if g is a stable equivalence, and f/τ induces isomorphisms in Margolis homologies if and only if g/τ induces isomorphisms in Margolis homologies. In other words, we may assume that f is surjective. (From a model categorical perspective, we have replaced f by an equivalent fibration.)

Let K be the kernel of f . The short exact sequence

$$0 \rightarrow K \rightarrow M \xrightarrow{f} N \rightarrow 0$$

induces a short exact sequence

$$0 \rightarrow K/\tau \rightarrow M/\tau \xrightarrow{f/\tau} N/\tau \rightarrow 0$$

by Proposition 3.1. This last short exact sequence induces long exact sequences in Margolis homologies with respect to Q_0 , Q_1 and Sq^2 . The long exact sequence shows that f/τ is an isomorphism in Margolis homologies if and only if K/τ has vanishing Margolis homologies. Finally, Proposition 4.7 implies that K/τ has vanishing Margolis homologies if and only if K is projective. Note that K is finitely generated and \mathbb{M}_2 -free because it is a subobject of the finitely generated \mathbb{M}_2 -free module M . Finally, K is projective if and only if f is a stable equivalence. \square

We establish a Künneth theorem for Margolis homology.

Proposition 4.10. *Let M and N be $\mathcal{A}(1)$ -modules that are free over \mathbb{M}_2 . Then*

$$H(M/\tau \otimes N/\tau; x) \cong H(M/\tau; x) \otimes H(N/\tau; x)$$

when x is Q_0 , Q_1 , or Sq^2 .

Proof. Lemma 4.2 gives the coproduct formula

$$\Delta(\text{Sq}^2) = \text{Sq}^2 \otimes 1 + \tau \text{Sq}^1 \otimes \text{Sq}^1 + 1 \otimes \text{Sq}^2.$$

Therefore, Sq^2 is primitive modulo τ . In particular, it acts as a derivation on $M/\tau \otimes N/\tau$. The isomorphism in Sq^2 -homology follows from the classical Künneth formula for chain complexes over \mathbb{F}_2 .

The arguments for Q_0 and Q_1 are the same, except slightly easier because these elements are primitive even before quotienting by τ . \square

Proposition 4.11. *Let M be a finitely generated $\mathcal{A}(1)$ -module that is \mathbb{M}_2 -free. Then M is invertible if and only if M/τ has one-dimensional Margolis homologies with respect to Q_0 , Q_1 , and Sq^2 .*

FIGURE 3. The $\mathcal{A}(1)$ -module J

Proof. First suppose that M is invertible. In other words, there exists an $\mathcal{A}(1)$ -module N and a stable equivalence

$$M \otimes N \xrightarrow{\simeq} \mathbb{M}_2.$$

Proposition 3.1 implies that there is a stable equivalence

$$(M \otimes N)/\tau \xrightarrow{\simeq} \mathbb{F}_2$$

of $\mathcal{A}(1)/\tau$ -modules. Corollary 4.9 shows that

$$H((M \otimes N)/\tau; x) \longrightarrow H(\mathbb{F}_2; x)$$

is an isomorphism when x is Q_0 , Q_1 , or Sq^2 . Now use Proposition 4.10 to deduce that $H(M/\tau; x) \otimes H(N/\tau; x)$ is isomorphic to \mathbb{F}_2 . It follows that $H(M/\tau; x)$ is one-dimensional.

Now assume that M/τ has one-dimensional Margolis homologies. Note that

$$H(D(M/\tau); x) \cong \text{Hom}_{\mathbb{F}_2}(H(M/\tau; x); \mathbb{F}_2)$$

when x is Q_0 , Q_1 , or Sq^2 . Therefore, $D(M/\tau)$ also has one-dimensional Margolis homologies. By Proposition 4.10, $M/\tau \otimes D(M/\tau)$ also has one-dimensional Margolis homologies. Hence the evaluation map

$$M/\tau \otimes D(M/\tau) \longrightarrow \mathbb{F}_2$$

induces an isomorphism in Margolis homologies because both sides are one-dimensional. Note that $M/\tau \otimes D(M/\tau)$ is isomorphic to $(M \otimes DM)/\tau$ by Proposition 3.1. Finally, Corollary 4.9 shows that the evaluation map

$$M \otimes DM \longrightarrow \mathbb{M}_2$$

is a stable equivalence. This shows that M is invertible with inverse DM . \square

5. THE PICARD GROUP OF $\mathcal{A}(1)$

Definition 5.1. Let J be the $\mathcal{A}(1)$ -module on two generators x and y of degrees $(0, 0)$ and $(2, 0)$ respectively, subject to the relations $\text{Sq}^2 x = \tau y$, $\text{Sq}^1 \text{Sq}^2 \text{Sq}^1 x = \text{Sq}^2 y$, and $\text{Sq}^1 y = 0$.

Figure 3 represents J as an $\mathcal{A}(1)$ -module.

Lemma 5.2. *The $\mathcal{A}(1)$ -module J is invertible, and the order of $[J]$ in $\text{Pic}(\mathcal{A}(1))$ is infinite.*

Proof. Proposition 4.11 implies that J is invertible. The Q_0 -homology and Q_1 -homology of J/τ are generated by x , while the Sq^2 -homology of J/τ is generated by y .

The degrees of x and y are different. Therefore, the Sq^2 -homology and the Q_0 -homology of any tensor power $J^{\otimes n}$ of J are in different degrees. On the other hand, the Sq^2 -homology and the Q_0 -homology of M_2 are in the same degree. This shows that $J^{\otimes n}$ is not stably equivalent to M_2 . \square

Remark 5.3. The classical joker is self-dual as an $\mathcal{A}(1)^{cl}$ -module. Therefore, it represents an element of order two in $\text{Pic}(\mathcal{A}(1)^{cl})$. On the other hand, Figure 3 shows that the motivic joker is not self-dual.

Theorem 5.4. *There is an isomorphism*

$$\mathbb{Z}^4 \longrightarrow \text{Pic}(\mathcal{A}(1))$$

sending (a, b, c, d) to the class of $\Sigma^{a,b}\Omega^c J^d$.

Proof. Recall the homomorphism

$$V : \text{Pic}(\mathcal{A}(1)) \longrightarrow \text{Pic}(A/\tau)$$

from Proposition 3.6. Consider the composition

$$\mathbb{Z}^4 \longrightarrow \text{Pic}(\mathcal{A}(1)) \xrightarrow{V} \text{Pic}(A/\tau) \xrightarrow{\cong} \text{Pic}(\mathbb{F}_2[D_8]),$$

where the last isomorphism comes from Lemma 4.3.

Recall from [CT00, Theorem 5.4] that the ungraded Picard group of $\mathbb{F}_2[D_8]$ is isomorphic to \mathbb{Z}^2 , generated by $\Omega\mathbb{F}_2$ and a module L . If we add the motivic bigrading, then we obtain that the graded Picard group $\text{Pic}(\mathbb{F}_2[D_8])$ is isomorphic to \mathbb{Z}^4 .

By direct computation, the composition sends the joker J to ΩL . Therefore, the composition is an isomorphism. This shows that V is surjective. We already know that V is injective from Proposition 3.6. Therefore, V is an isomorphism, so the map

$$\mathbb{Z}^4 \longrightarrow \text{Pic}(\mathcal{A}(1))$$

is an isomorphism as well. \square

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